Int. J. Solids Structures Vol. 29, No. 8, pp. 1019–1032, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

PARAMETRIC EXCITATION UNDER MULTIPLE EXCITATION PARAMETERS: ASYMMETRIC PLATES UNDER A ROTATING SPRING

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(*Received* 27 *December* 1990; *in revised form* 15 *July* 1991)

Abstract—A perturbation method is developed to predict stability of parametrically excited dynamic systems containing multiple perturbation parameters. This method, based on the Floquet theorem and the method of successive approximations, results in a non-linear matrix eigenvalue problem whose eigenvalues are used to predict the system stability. The method is applied to a classical circular plate, containing elastic or viscoelastic inclusions, excited by a linear transverse spring rotating at constant speed. Primary and secondary resonances are predicted. The transition to instability predicted by the perturbation analysis agrees with predictions obtained by numerical integration of the equations of motion.

1. INTRODUCTION

Parametrically excited systems usually contain a single excitation parameter κ and take a standard form

$$\ddot{\mathbf{q}}+\kappa\sum_{s=-\infty}^{\infty}\mathbf{C}^{(s)}\,e^{is\Omega t}\,\dot{\mathbf{q}}+\left[\mathbf{B}+\kappa\sum_{s=-\infty}^{\infty}\mathbf{H}^{(s)}\,e^{is\Omega t}\right]\mathbf{q}=0 \tag{1}$$

where $\mathbf{q}$ is a column vector, $\mathbf{B}$ is a diagonal matrix and $\mathbf{H}^{(s)}$ and $\mathbf{C}^{(s)}$, $s = 0, \pm 1, \pm 2, \ldots$, are square matrices. When κ is small, perturbation methods such as the method of multiple scales (Nayfeh and Mook, 1979) and the Krylov–Bogoliubov–Mitropolsky method can be used to determine $\mathbf{q}$ analytically. Perturbation analysis shows that (1) possesses unbounded response at particular values of Ω (Valeev, 1963; Hsu, 1963, 1965; Nayfeh and Mook, 1979). Elimination of secular terms evolving in the perturbation analysis gives the transition curves identifying the stability boundaries in the κ–Ω plane.

Parametrically excited systems with multiple excitation parameters also arise in engineering. For example, circular saws with viscoelastic material placed in radial rim slots have been used to inhibit thermal buckling of the plate and simultaneously increase its damping without increasing its thickness. A rotating asymmetric saw, excited by stationary guide bearings and the workpiece, can be modeled as a parametrically excited system with two excitation parameters: the normalized stiffness κ of the guide bearing and the dimensionless measure of the inclusion size ε (Shen and Mote, 1991b). The response of such systems satisfies

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$$\ddot{\mathbf{q}} + \sum_{s=-r}^{r} \mathbf{C}^{(s)}(\varepsilon)\, e^{is\Omega t}\, \dot{\mathbf{q}} + \left[\mathbf{B}(\varepsilon) + \kappa \sum_{s=-\zeta}^{\zeta} \mathbf{H}^{(s)}(\varepsilon)\, e^{is\Omega t} \right] \mathbf{q} = 0 \tag{2}$$

where $\mathbf{C}^{(s)}(\varepsilon)$, $\mathbf{B}(\varepsilon)$ and $\mathbf{H}^{(s)}(\varepsilon)$ are convergent perturbation series in ε with

$$\mathbf{C}^{(s)}(\varepsilon) \equiv [C_{ij}^{(s)}(\varepsilon)] = \mathbf{C}^{(s)}(\varepsilon)^{\mathrm{T}} = \begin{cases} \varepsilon \mathbf{C}_1 + \varepsilon^2 \mathbf{C}_2 + \cdots, & s = 0 \\ \mathbf{0}, & s \neq 0 \end{cases} \tag{3a}$$

$$\mathbf{B}(\varepsilon) \equiv \mathrm{diag}\,[\omega_1^2(\varepsilon), \omega_2^2(\varepsilon), \ldots] = \mathbf{B}_0 + \varepsilon \mathbf{B}_1 + \varepsilon^2 \mathbf{B}_2 + \cdots \tag{3b}$$

$$\mathbf{H}^{(s)}(\varepsilon) \equiv [H_{ij}^{(s)}(\varepsilon)] = \mathbf{H}^{(s)}(\varepsilon)^{\mathrm{T}} = \bar{\mathbf{H}}^{(-s)}(\varepsilon) = \mathbf{H}_0^{(s)} + \varepsilon \mathbf{H}_1^{(s)} + \varepsilon^2 \mathbf{H}_2^{(s)} + \cdots \tag{3c}$$

One solution method specifies the relative order between κ and ε *a priori* (e.g. $\kappa \approx \varepsilon^2$) to transform (2) into a system with one perturbation parameter. Application of the existing perturbation methods can then determine system stability for any relative order specified. The process becomes unwieldy and arbitrary for several perturbation parameters and relative orders.

The purpose of this paper is to present a new perturbation method for parametrically excited systems that does not require *a priori* specification of relative order among multiple perturbation parameters. In this method, the response is represented as a product of a characteristic exponential and a Fourier series following the Floquet theorem. Then application of the method of successive approximation (Valeev, 1963) results in a non-linear matrix eigenvalue problem whose eigenvalues give the characteristic exponents and stability transition curves of the system. This method is applied to a classical, circular plate with small elastic or viscoelastic inclusions to determine system stability and transition curves when the plate is excited by a linear, transverse spring rotating at constant speed.

2. STABILITY ANALYSIS

Consider an N degree of freedom parametrically excited system governed by (2) and (3a–c) with α characteristic frequencies

$$\omega_i = \omega_j + \mu_i, \quad i = j, j+1, \ldots, j+\alpha-1$$

near ω_j and β characteristic frequencies

$$\omega_i = \omega_k + \mu_i, \quad i = k, k+1, \ldots, k+\beta-1$$

near ω_k, in which μ_i is small compared with both ω_j and ω_k. According to the Floquet theorem,

$$\mathbf{q}(t) = e^{pt} \sum_{n=-\infty}^{\infty} \mathbf{u}^{(n)}\, e^{in\Omega t} \tag{4}$$

where $p = p(\kappa, \varepsilon)$ is a characteristic exponent. Substitute (4) into (2) and collect terms to obtain

$$[(p + in\Omega)^2 \mathbf{I}_N + \mathbf{B}]\mathbf{u}^{(n)} + \sum_{s-n=-r}^{r} [(p + is\Omega)\mathbf{C}^{(n-s)} + \kappa \mathbf{H}^{(n-s)}]\mathbf{u}^{(s)} = \mathbf{0}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{5}$$

where $\mathbf{I}_N$ is an $N \times N$ identity matrix and $\mathbf{u}^{(n)} = (u_1^{(n)}, \ldots, u_N^{(n)})^{\mathrm{T}}$. The equations in (5) are

$$e_r^{(n)} u_r^{(n)} + \sum_{s=-r}^{r} \sum_{q=1}^{N} f_{rq}^{(n,s)} u_q^{(s)} = 0, \quad n = 0, \pm 1, \pm 2, \ldots, \quad r = 1, 2, \ldots, N \tag{6}$$

where

$$e_r^{(n)} = e_r^{(n)}(\Omega, p) = (p+in\Omega)^2+\omega_r^2 \tag{7a}$$

$$f_{rq}^{(n,s)} = (p+is\Omega)C_{rq}^{(n-s)}+\kappa H_{rq}^{(n-s)}. \tag{7b}$$

Consider a characteristic exponent $p(\kappa, \varepsilon) \approx p_0 \equiv i\omega_j$ at a particular combination resonance $\Omega = \Omega_0 \equiv (\omega_j+\omega_k)/l$. For such (Ω_0, p_0),

$$e_r^{(n)}(\Omega_0, p_0) = 0, \quad \text{if} \quad (n,r) \in Y_{jk}^{(0,-l)} \tag{8}$$

where

$$Y_{jk}^{(0,-l)} \equiv \{(0,j),(0,j+1),\ldots,(0,j+\alpha-1)\} \cup \{(-l,k),(-l,k+1),\ldots,(-l,k+\beta-1)\}. \tag{9}$$

Equations (8), (9) and the definition (7a) imply that

$$e_r^{(n)}(\Omega, p) \sim \begin{cases} O(\kappa) \quad \text{or} \quad O(\varepsilon), & \text{if} \quad (n,r) \in Y_{jk}^{(0,-l)} \\ O(1), & \text{if} \quad (n,r) \notin Y_{jk}^{(0,-l)} \end{cases}.$$

Therefore, (6) can be rearranged as

$$u_r^{(n)} = -d_r^{(n)} \sum_{s,q}' f_{rq}^{(n,s)} u_q^{(s)} - d_r^{(n)} \sum_{q=j}^{j+\alpha-1} f_{rq}^{(n,0)} u_q^{(0)} - d_r^{(n)} \sum_{q=k}^{k+\beta-1} f_{rq}^{(n,-l)} u_q^{(-l)} \tag{10a}$$

for $(n,r) \notin Y_{jk}^{(0,-l)}$, and

$$e_r^{(n)}(\Omega, p) u_r^{(n)} = -\sum_{s,q}' f_{rq}^{(n,s)} u_q^{(s)} - \sum_{q=j}^{j+\alpha-1} f_{rq}^{(n,0)} u_q^{(0)} - \sum_{q=k}^{k+\beta-1} f_{rq}^{(n,-l)} u_q^{(-l)} \tag{10b}$$

for $(n,r) \in Y_{jk}^{(0,-l)}$, where $d_r^{(n)} = [e_r^{(n)}(\Omega,p)]^{-1} \sim O(1)$ and $\sum_{s,q}' = \sum_{s=-\infty}^{\infty} \sum_{q=1}^{N}$ with $(s,q) \notin Y_{jk}^{(0,-l)}$.

It can be shown that (10a) is a contraction mapping in an L_1 norm $\|\mathbf{u}\|_1 \equiv \sum_{s,q}' \left|u_q^{(s)}\right|$ if ε and κ are small enough such that the contraction constant

$$\mu \equiv \max \left\{ \sum_{n,r}' \left| d_r^{(n)} f_{rq}^{(n,s)} \right|; \quad -\infty \leqslant s \leqslant \infty, 1 \leqslant q \leqslant N, (s,q) \notin Y_{jk}^{(0,-l)} \right\}$$

satisfies $\mu < 1$. Therefore, $\mu_r^{(n)}, (n,r) \notin Y_{jk}^{(0,-l)}$ in (10a) can be determined in terms of $u_r^{(n)}, (n,r) \in Y_{jk}^{(0,-l)}$ by successive approximation to any precision independent of the relative orders of κ and ε. The numer of iterations needed depends on the order of the perturbation analysis. For second order perturbation analysis, second iteration of the contraction mapping (10a), with the initial condition $u_q^{(s)} = 0$ for $(s,q) \notin Y_{jk}^{(0,-l)}$, results in

$$[u_r^{(n)}]^{\langle 2 \rangle} = d_r^{(n)}(\Omega,p) \sum_{s,q}' f_{rq}^{(n,s)} d_q^{(s)}(\Omega,p) \left[\sum_{m=j}^{j+\alpha-1} f_{qm}^{(s,0)} u_m^{(0)} + \sum_{m=k}^{k+\beta-1} f_{qm}^{(s,-l)} u_m^{(-l)} \right]$$
$$- d_r^{(n)}(\Omega,p) \left[\sum_{m=j}^{j+\alpha-1} f_{rm}^{(n,0)} u_m^{(0)} + \sum_{m=k}^{k+\beta-1} f_{rm}^{(n,-l)} u_m^{(-l)} \right], \quad (n,r) \notin Y_{jk}^{(0,-l)}.$$

Substitute $[u_r^{(n)}]^{\langle 2 \rangle}$ for $u_q^{(s)}$ in (10b) and retain terms up to second degree in $f_{rq}^{(n,s)}$ to obtain

$$\begin{bmatrix} \mathbf{E}_{\mathbf{j}}^{(0)}(\Omega, p) & \mathbf{0} \\ \mathbf{0} & -\mathbf{E}_{\mathbf{k}}^{(-l)}(\Omega, p) \end{bmatrix} \begin{pmatrix} \mathbf{v}_{\mathbf{j}}^{(0)} \\ \mathbf{v}_{\mathbf{k}}^{(-l)} \end{pmatrix} = \begin{bmatrix} \mathbf{B}_{\mathbf{jj}}^{(0,0)} & i\mathbf{B}_{\mathbf{jk}}^{(0,-l)} \\ i\mathbf{B}_{\mathbf{kj}}^{(-l,0)} & -\mathbf{B}_{\mathbf{kk}}^{(-l,-l)} \end{bmatrix} \begin{pmatrix} \mathbf{v}_{\mathbf{j}}^{(0)} \\ \mathbf{v}_{\mathbf{k}}^{(-l)} \end{pmatrix} \quad (11)$$

where boldface indices **j** and **k** in (11) denote the index ranges $\mathbf{j} = \{j, j+1, \ldots, j+\alpha-1\}$ and $\mathbf{k} = \{k, k+1, \ldots, k+\beta-1\}$. Also,

$$\mathbf{v}_{\mathbf{j}}^{(0)} = \sqrt{2\omega_{\mathbf{j}}}(u_j^{(0)}, u_{j+1}^{(0)}, \ldots, u_{j+\alpha-1}^{(0)})^{\mathrm{T}}$$

$$\mathbf{v}_{\mathbf{k}}^{(-l)} = -i\sqrt{2\omega_{\mathbf{k}}}(u_k^{(-l)}, u_{k+1}^{(-l)}, \ldots, u_{k+\beta-1}^{(-l)})^{\mathrm{T}}$$

$$\mathbf{E}_{\mathbf{j}}^{(0)}(\Omega, p) = \frac{1}{2\omega_{\mathbf{j}}} \operatorname{diag}[e_j^{(0)}(\Omega, p), e_{j+1}^{(0)}(\Omega, p), \ldots, e_{j+\alpha-1}^{(0)}(\Omega, p)]$$

$$\mathbf{E}_{\mathbf{k}}^{(-l)}(\Omega, p) = \frac{1}{2\omega_{\mathbf{k}}} \operatorname{diag}[e_k^{(-l)}(\Omega, p), e_{k+1}^{(-l)}(\Omega, p), \ldots, e_{k+\beta-1}^{(-l)}(\Omega, p)]$$

and

$$B_{\mathbf{jk}}^{(m,n)} = \frac{1}{2\sqrt{\omega_{\mathbf{j}}\omega_{\mathbf{k}}}}[B_{\gamma\delta}^{(m,n)}]_{\alpha\times\beta}, \quad \begin{matrix} \gamma = j, j+1, \ldots, j+\alpha-1 \in \mathbf{j} \\ \delta = k, k+1, \ldots, k+\beta-1 \in \mathbf{k} \end{matrix}$$

with

$$B_{\gamma\delta}^{(m,n)} = \sum_{s,q}{}' f_{\gamma q}^{(m,s)} d_q^{(s)}(\Omega, p) f_{q\delta}^{(s,n)} - f_{\gamma\delta}^{(m,n)}.$$

Introduce two detunings z and σ defined by

$$p = i\omega_{\mathbf{j}} + iz, \quad l\Omega = \omega_{\mathbf{j}} + \omega_{\mathbf{k}} + \sigma. \quad (12)$$

The existence of non-trivial solutions of (11) allows the determination of z in terms of σ. Substitute (12) and (7a,b) into (11) and recall $\mathbf{C}^{(s)} = \mathbf{0}$ for $s \neq 0$ to reduce (11) to a non-linear matrix eigenvalue problem

$$(i\mathbf{C}^{\langle 0\rangle} + \mathbf{C}^{\langle 1\rangle} + \Sigma(\sigma) - z\mathbf{I}_{\alpha+\beta})\mathbf{v} + [\mathbf{R}(z, \sigma) + \kappa(\mathbf{K}^{\langle 0\rangle} - \mathbf{K}^{\langle 2\rangle}) - \kappa^2\mathbf{K}^{\langle 1\rangle}]\mathbf{v} = 0 \quad (13a)$$

where z is the eigenvalue for specified σ,

$$\mathbf{C}^{\langle 0\rangle} = \begin{bmatrix} \mathbf{C}_{\mathbf{jj}}^{\langle 0\rangle} & 0 \\ 0 & \mathbf{C}_{\mathbf{kk}}^{\langle 0\rangle} \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{2}[C_{ab}^{(0)}]_{\alpha\times\alpha} & 0 \\ 0 & \frac{1}{2}[C_{cd}^{(0)}]_{\beta\times\beta} \end{bmatrix} \quad (13b)$$

$$\mathbf{C}^{\langle 1\rangle} = \begin{bmatrix} \mathbf{C}_{\mathbf{jj}}^{\langle 1\rangle} & 0 \\ 0 & \mathbf{C}_{\mathbf{kk}}^{\langle 1\rangle} \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{2}\omega_{\mathbf{j}}\left[\sum_{\substack{q=1 \\ q\notin\mathbf{j}}}^{N} C_{aq}^{(0)} d_q^{(0)} C_{qb}^{(0)}\right]_{\alpha\times\alpha} & 0 \\ 0 & -\frac{1}{2}\omega_{\mathbf{k}}\left[\sum_{\substack{q=1 \\ q\notin\mathbf{k}}}^{N} C_{cq}^{(0)} d_q^{(0)} C_{qd}^{(0)}\right]_{\beta\times\beta} \end{bmatrix} \quad (13c)$$

$$\Sigma = \begin{bmatrix} \Delta_{\mathbf{j}} & 0 \\ \mathbf{0} & \sigma\mathbf{I}_\beta - \Delta_{\mathbf{k}} \end{bmatrix}, \quad \begin{matrix} \Delta_{\mathbf{j}} = \operatorname{diag}[\mu_j, \mu_{j+1}, \ldots, \mu_{j+\alpha-1}] \\ \Delta_{\mathbf{k}} = \operatorname{diag}[\mu_k, \mu_{k+1}, \ldots, \mu_{k+\beta-1}] \end{matrix} \quad (13d)$$

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_{\mathbf{j}}^{(0)} \\ \mathbf{v}_{\mathbf{k}}^{(-l)} \end{pmatrix}, \quad \mathbf{R}(z) = \begin{bmatrix} -\frac{1}{2\omega_{\mathbf{j}}}(z^2\mathbf{I}_\alpha - \mathbf{\Delta}_{\mathbf{j}}^2) & \mathbf{0} \\ \mathbf{0} & \frac{1}{2\omega_{\mathbf{k}}}[(z-\sigma)^2\mathbf{I}_\beta - \mathbf{\Delta}_{\mathbf{k}}^2] \end{bmatrix} \tag{13e}$$

$$\mathbf{K}^{\langle 0\rangle} = \begin{bmatrix} \mathbf{K}_{\mathbf{jj}}^{\langle 0\rangle} & i\mathbf{K}_{\mathbf{jk}(l)}^{\langle 0\rangle} \\ i\mathbf{K}_{\mathbf{kj}(-l)}^{\langle 0\rangle} & -\mathbf{K}_{\mathbf{kk}}^{\langle 0\rangle} \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{2\omega_{\mathbf{j}}}[H_{ab}^{(0)}]_{\alpha\times\alpha} & \frac{i}{2\sqrt{\omega_{\mathbf{j}}\omega_{\mathbf{k}}}}[H_{ad}^{(l)}]_{\alpha\times\beta} \\ \frac{i}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{j}}}}[H_{cb}^{(-l)}]_{\beta\times\alpha} & -\frac{1}{2\omega_{\mathbf{k}}}[H_{cd}^{(0)}]_{\beta\times\beta} \end{bmatrix} \tag{13f}$$

$$\mathbf{K}^{\langle 1\rangle} = \begin{bmatrix} \mathbf{K}_{\mathbf{jj}(0)}^{\langle 1\rangle} & i\mathbf{K}_{\mathbf{jk}(l)}^{\langle 1\rangle} \\ i\mathbf{K}_{\mathbf{kj}(-l)}^{\langle 1\rangle} & -\mathbf{K}_{\mathbf{kk}(l)}^{\langle 1\rangle} \end{bmatrix}$$

$$\equiv \begin{bmatrix} \frac{1}{2\omega_{\mathbf{j}}}\left[\sum_{s,q}' H_{aq}^{(-s)}d_q^{(s)}H_{qb}^{(s)}\right]_{\alpha\times\alpha} & \frac{i}{2\sqrt{\omega_{\mathbf{j}}\omega_{\mathbf{k}}}}\left[\sum_{s,q}' H_{aq}^{(-s)}d_q^{(s)}H_{qd}^{(s+l)}\right]_{\alpha\times\beta} \\ \frac{i}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{j}}}}\left[\sum_{s,q}' H_{cq}^{(-l-s)}d_q^{(s)}H_{qb}^{(s)}\right]_{\beta\times\alpha} & -\frac{1}{2\omega_{\mathbf{k}}}\left[\sum_{s,q}' H_{cq}^{(-l-s)}d_q^{(s)}H_{qd}^{(s+l)}\right]_{\beta\times\beta} \end{bmatrix} \tag{13g}$$

$$\mathbf{K}^{\langle 2\rangle} = \begin{bmatrix} i\mathbf{K}_{\mathbf{jj}(0)}^{\langle 2\rangle} & \mathbf{K}_{\mathbf{jk}(l)}^{\langle 2\rangle} \\ \mathbf{K}_{\mathbf{kj}(-l)}^{\langle 2\rangle} & i\mathbf{K}_{\mathbf{kk}(l)}^{\langle 2\rangle} \end{bmatrix}, \tag{13h}$$

with

$$\mathbf{K}_{\mathbf{jj}(0)}^{\langle 2\rangle} \equiv \frac{1}{2}\left[\sum_{\substack{q=1\\q\notin\mathbf{j}}}^{N}(C_{aq}^{(0)}d_q^{(0)}H_{qb}^{(0)} + H_{aq}^{(0)}d_q^{(0)}C_{qb}^{(0)})\right]_{\alpha\times\alpha}$$

$$\mathbf{K}_{\mathbf{kj}(-l)}^{\langle 2\rangle} \equiv -\frac{1}{2}\sqrt{\frac{\omega_{\mathbf{j}}}{\omega_{\mathbf{k}}}}\left[\sum_{\substack{q=1\\q\notin\mathbf{j}}}^{N}H_{cq}^{(-l)}d_q^{(0)}C_{qb}^{(0)}\right]_{\beta\times\alpha} + \frac{1}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{j}}}}\left[\sum_{\substack{q=1\\q\notin\mathbf{k}}}^{N}C_{cq}^{(0)}d_q^{(-l)}H_{qb}^{(-l)}\right]_{\beta\times\alpha}$$

$$\mathbf{K}_{\mathbf{jk}(l)}^{\langle 2\rangle} \equiv -\frac{1}{2}\sqrt{\frac{\omega_{\mathbf{j}}}{\omega_{\mathbf{k}}}}\left[\sum_{\substack{q=1\\q\notin\mathbf{j}}}^{N}C_{aq}^{(0)}d_q^{(0)}H_{qd}^{(l)}\right]_{\alpha\times\beta} + \frac{1}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{j}}}}\left[\sum_{\substack{q=1\\q\notin\mathbf{k}}}^{N}H_{aq}^{(l)}d_q^{(-l)}C_{qd}^{(0)}\right]_{\alpha\times\beta}$$

$$\mathbf{K}_{\mathbf{kk}(l)}^{\langle 2\rangle} = \frac{1}{2}\left[\sum_{\substack{q=1\\q\notin\mathbf{k}}}^{N}(C_{cq}^{(0)}d_q^{(-l)}H_{qd}^{(0)} + H_{cq}^{(0)}d_q^{(-l)}C_{qd}^{(0)})\right]_{\beta\times\beta}$$

with indices

$$a, b = j, j+1, \ldots, j+\alpha-1 \in \mathbf{j}$$
$$c, d = k, k+1, \ldots, k+\beta-1 \in \mathbf{k}$$

and symmetry

$$\mathbf{C}^{\langle 0\rangle} = \mathbf{C}^{\langle 0\rangle\mathrm{T}}, \quad \mathbf{C}^{\langle 1\rangle} = \mathbf{C}^{\langle 1\rangle\mathrm{T}}$$

$$\mathbf{K}_{\mathbf{jj}(\cdot)}^{\langle i\rangle\mathrm{T}} = \bar{\mathbf{K}}_{\mathbf{jj}(\cdot)}^{\langle i\rangle}, \quad \mathbf{K}_{\mathbf{kk}(\cdot)}^{\langle i\rangle\mathrm{T}} = \bar{\mathbf{K}}_{\mathbf{kk}(\cdot)}^{\langle i\rangle}, \quad \mathbf{K}_{\mathbf{jk}(l)}^{\langle i\rangle\mathrm{T}} = \bar{\mathbf{K}}_{\mathbf{kj}(-l)}^{\langle i\rangle}, \quad i = 0, 1, 2.$$

The eigenvalue z for specified σ in (13a) can be solved either numerically or through perturbation as illustrated in the next section. In addition, (13a) is valid for any relative orders of κ and ε as long as κ and ε are sufficiently small that (10a) is a contraction mapping. The system is asymptotically stable if $\text{Im}[z] > 0$. A similar procedure applies for higher order perturbations or systems containing additional perturbation parameters.

3. SPECIAL CASES

The stability of $\mathbf{q}$ can be determined from (13a) through perturbation if a relative order of κ and ε is specified. Three cases are examined. In each case the stability of the undamped system is studied first, and then damping of order ε is introduced. These cases arise in asymmetric plates under a rotating spring with coefficient κ.

3.1. *Primary resonances of $O(\kappa)$*

This instability occurs when $\kappa \ll \varepsilon$, $\mu_i \sim O(\kappa)$, $i \in \mathbf{j}, \mathbf{k}$, $\mathbf{K}^{\langle 0\rangle} \sim O(1)$, $p = i\omega_j + iz$, $l\Omega = \omega_j + \omega_k + \sigma$, and

$$\sigma \sim O(\kappa). \tag{14}$$

Undamped systems. $\mathbf{C}^{\langle 0\rangle} = 0$. Let

$$z = \kappa z^* + \kappa^2 z^{**} + \cdots \tag{15}$$

and substitute (14) and (15) into (13a) to obtain

$$(\mathbf{U}_1 - \kappa z^* \mathbf{I}_{\alpha+\beta})\mathbf{v} + O(\kappa^2) = 0 \tag{16}$$

where $\mathbf{U}_1 \equiv \mathbf{D}_1 + i\mathbf{D}_2$ with

$$\mathbf{D}_1 = \bar{\mathbf{D}}_1^T = \begin{bmatrix} \Delta_j + \kappa \mathbf{K}_{jj}^{\langle 0\rangle} & \mathbf{0} \\ \mathbf{0} & \sigma \mathbf{I}_\beta - \Delta_k - \kappa \mathbf{K}_{kk}^{\langle 0\rangle} \end{bmatrix} \tag{17a}$$

$$\mathbf{D}_2 = \bar{\mathbf{D}}_2^T = \begin{bmatrix} \mathbf{0} & \kappa \mathbf{K}_{jk(l)}^{\langle 0\rangle} \\ \kappa \mathbf{K}_{kj(-l)}^{\langle 0\rangle} & \mathbf{0} \end{bmatrix}. \tag{17b}$$

Stability is predicted by the eigenvalue κz^* of $\mathbf{U}_1$. If $\text{Im}[\kappa z^*] > 0$, then the system is asymptotically stable up to $O(\kappa)$. If $\text{Im}[\kappa z^*] < 0$, the system is unstable. If $\text{Im}[\kappa z^*] = 0$, the system stability may be determined by $O(\kappa^2)$.

Damped systems. $\mathbf{C}^{\langle 0\rangle} \sim O(\varepsilon)$. Substitute (14) and (15) into (13a) to obtain

$$(\mathbf{U}_2 - \kappa z^* \mathbf{I}_{\alpha+\beta})\mathbf{v} + O(\kappa^2) = 0 \tag{18}$$

where $\mathbf{U}_2 \equiv \mathbf{U}_1 + i\mathbf{C}^{\langle 0\rangle}$. Therefore,

$$\text{Im}[\kappa z^*] = \text{Im}\left[\frac{\bar{\mathbf{u}}^T \mathbf{U}_2 \mathbf{u}}{\bar{\mathbf{u}}^T \mathbf{u}}\right] = \frac{1}{\bar{\mathbf{u}}^T \mathbf{u}}[\bar{\mathbf{u}}^T \mathbf{C}^{\langle 0\rangle} \mathbf{u} + \bar{\mathbf{u}}^T \mathbf{D}_2 \mathbf{u}] \tag{19}$$

where $\mathbf{u}$ is the eigenvector corresponding to κz^*. If $\mathbf{C}^{\langle 0\rangle}$ is positive definite, then $\bar{\mathbf{u}}^T \mathbf{C}^{\langle 0\rangle} \mathbf{u} > 0$. In addition, $\mathbf{C}^{\langle 0\rangle} \sim O(\varepsilon)$ and $\mathbf{D}_2 \sim O(\kappa)$ imply that $\bar{\mathbf{u}}^T \mathbf{C}^{\langle 0\rangle} \mathbf{u} \sim O(\varepsilon)$ and $\bar{\mathbf{u}}^T \mathbf{D}_2 \mathbf{u} \sim O(\kappa)$. Since $\kappa \ll \varepsilon$, (19) implies that $\text{Im}[\kappa z^*] > 0$. Positive definite damping $O(\varepsilon)$ stabilizes the undamped system asymptotically for κ up to $O(\varepsilon)$.

3.2. *Primary resonances of* $O(\varepsilon)$

This instability occurs when $\kappa \sim O(\varepsilon)$, $\mu_i \sim O(\varepsilon)$, $i \in \mathbf{j}, \mathbf{k}$, $\mathbf{K}^{\langle 0 \rangle} \sim O(1)$, $p = i\omega_j + iz$, $l\Omega = \omega_j + \omega_k + \sigma$, and

$$\sigma \sim O(\varepsilon). \tag{20}$$

Substitute (15) and (20) into (13a) and recall $\kappa \sim O(\varepsilon)$ to obtain (16) for undamped systems and (18) for damped systems. Since $\kappa \sim O(\varepsilon)$, (19) can result in negative $\mathrm{Im}[\kappa z^*]$. Positive damping $C^{\langle 0 \rangle}$ does not necessarily stabilize the undamped system. A sufficient condition for stability requires $\mathbf{D}_2 + \mathbf{C}^{\langle 0 \rangle}$ positive definite.

3.3. *Secondary resonance of* $O(\kappa^2)$

This resonance occurs when $\kappa \sim O(\varepsilon)$, $\mathbf{K}^{\langle 0 \rangle}_{\mathbf{jk}(l)} \sim O(\varepsilon)$, $\mathbf{K}^{\langle 0 \rangle}_{\mathbf{jj}} = h_j \mathbf{I}_\alpha$, $\mathbf{K}^{\langle 0 \rangle}_{\mathbf{kk}} = h_k \mathbf{I}_\beta$, h_j, $h_k \sim O(1)$, $\mathbf{K}^{\langle 1 \rangle} \sim O(1)$, $\Delta_j = \mu_j \mathbf{I}_\alpha$, $\Delta_k = \mu_k \mathbf{I}_\beta$, μ_j, $\mu_k \sim O(\varepsilon)$, $p = i\omega_j + iz$, $l\Omega = \omega_j + \omega_k + \sigma$.

Undamped systems. Let

$$\sigma = \sigma_0 + \kappa^2 \sigma^{**} + \cdots \tag{21a}$$

$$z = z_0 + \kappa^2 z^{**} + \cdots \tag{21b}$$

where σ_0 and z_0 contain only first order terms of κ and ε. Substitute (21a,b) into (13a) to get

$$(\mathbf{U}_3 + \mathbf{U}_0 - \kappa^2 z^{**})\mathbf{v} + O(\kappa^3) = \mathbf{0} \tag{22}$$

where $\mathbf{U}_3 = \mathbf{D}_3 + i\mathbf{D}_4$ with

$$\mathbf{D}_3 = \bar{\mathbf{D}}_3^T = \begin{bmatrix} -\dfrac{\kappa h_j}{2\omega_j}(\kappa h_j + 2\mu_j)\mathbf{I}_\alpha - \kappa^2 \mathbf{K}^{\langle 1 \rangle}_{\mathbf{jj}(0)} & \mathbf{0} \\ \mathbf{0} & \kappa^2 \sigma^{**} \mathbf{I}_\beta + \dfrac{\kappa h_k}{2\omega_k}(\kappa h_k + 2\mu_k)\mathbf{I}_\beta + \kappa^2 \mathbf{K}^{\langle 1 \rangle}_{\mathbf{kk}(l)} \end{bmatrix}$$

$$\mathbf{D}_4 = \bar{\mathbf{D}}_4^T = \begin{bmatrix} \mathbf{0} & \kappa \mathbf{K}^{\langle 0 \rangle}_{\mathbf{jk}(l)} - \kappa^2 \mathbf{K}^{\langle 1 \rangle}_{\mathbf{jk}(l)} \\ \kappa \mathbf{K}^{\langle 0 \rangle}_{\mathbf{kj}(-l)} - \kappa^2 \mathbf{K}^{\langle 1 \rangle}_{\mathbf{kj}(-l)} & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{U}_0 = \begin{bmatrix} (\mu_j - z_0 + \kappa h_j)\mathbf{I}_\alpha & \mathbf{0} \\ \mathbf{0} & (\sigma_0 - z_0 - \mu_k - \kappa h_k)\mathbf{I}_\beta \end{bmatrix}.$$

Secondary resonance occurs when $\mathbf{U}_0 = 0$ and $\mathrm{Im}[\kappa^2 z^{**}] < 0$. This implies that

$$\sigma_0 = \kappa(h_j + h_k) + \mu_j + \mu_k$$

$$z_0 = \kappa h_j + \mu_j.$$

Therefore, the secondary instability region in the $\kappa - \Omega$ plane surrounds the line

$$l\Omega = \kappa(h_j + h_k) + \mu_j + \mu_k + \omega_j + \omega_k. \tag{23}$$

Damped systems. $\mathbf{C}^{\langle 0 \rangle} \sim O(\varepsilon)$. Substitute (14) and (15) into (13a) and recall $\kappa \sim O(\varepsilon)$ as well as $\mathbf{K}^{\langle 0 \rangle}_{\mathbf{jk}(l)} \sim O(\varepsilon)$ to obtain (16) except that $\mathbf{U}_1$ is replaced by $\mathbf{U}_4 \equiv \mathbf{D}_5 + i\mathbf{C}^{\langle 0 \rangle}$, where

$$\mathbf{D}_5 = \bar{\mathbf{D}}_5^T = \begin{bmatrix} \Delta_j + \kappa \mathbf{K}_{jj}^{(0)} & 0 \\ 0 & \kappa\sigma^* \mathbf{I}_{\beta} - \Delta_k - \kappa \mathbf{K}_{kk}^{(0)} \end{bmatrix}. \tag{24}$$

Damping always stabilizes the system at least up to $O(\kappa)$ for positive definite $\mathbf{C}^{(0)}$, because

$$\mathrm{Im}\,[\kappa z^*] = \mathrm{Im}\left[\frac{\bar{\mathbf{u}}^T \mathbf{U}_4 \mathbf{u}}{\bar{\mathbf{u}}^T \mathbf{u}}\right] = \frac{\bar{\mathbf{u}}^T \mathbf{C}^{(0)} \mathbf{u}}{\bar{\mathbf{u}}^T \mathbf{u}} > 0$$

where $\mathbf{u}$ is the eigenvector corresponding to the eigenvalue κz^*.

For $\kappa^2 \sim O(\varepsilon)$ the resulting equation would be (21a,b) and (22) with $\mathbf{U}_3$ replaced by

$$\mathbf{U}_5 \equiv \mathbf{U}_3 + i\mathbf{C}^{(0)} = \mathbf{D}_3 + i(\mathbf{D}_4 + \mathbf{C}^{(0)}). \tag{25}$$

A sufficient condition for stability requires $\mathbf{D}_4 + \mathbf{C}^{(0)}$ positive definite.

4. CIRCULAR PLATES WITH SMALL INCLUSIONS

The perturbation method is applied to a circular plate containing small elastic or viscoelastic inclusions to predict its stability under excitation by a linear, transverse spring rotating at constant speed. The transition curves in the κ–Ω plane are compared to those obtained from numerical integration of (2).

4.1. *Theoretical background*

Let ε be a small, dimensionless measure of the inclusion size (e.g. ratio of the areas of the inclusions and the plate). The normalized eigensolutions $\psi_{mn}(\mathbf{r}), \beta_{mn}$ of the asymmetric plate, represented in terms of normalized eigensolutions $\phi_{ij}(\mathbf{r}), \omega_{ij}$ of the corresponding axisymmetric plate†, are

$$\beta_{mn}^2 = \omega_{mn}^2 + \varepsilon\mu_{mn}^{(1)} + \varepsilon^2\mu_{mn}^{(2)} + \cdots \tag{26a}$$

and

$$\psi_{mn}(\mathbf{r}) = \phi_{mn}(\mathbf{r}) + \sum_{i=0}^{L} \sum_{j=-L}^{L} [\varepsilon a_{mnij}^{(1)} + \varepsilon^2 a_{mnij}^{(2)} + \cdots]\phi_{ij}(\mathbf{r}) \tag{26b}$$

where $\mu_{mn}^{(1)}$, $\mu_{mn}^{(2)}$ and $a_{mnij}^{(1)}$ have been determined analytically by Shen and Mote (1991a). In addition, some normalized eigenvalue pairs β_{mn} and $\beta_{m,-n}$, that are repeated ($\omega_{mn} = \omega_{m,-n}$) in the axisymmetric plate, are distinct (split modes) in the asymmetric plate with

$$\Delta\beta_{mn} \equiv \beta_{mn} - \beta_{m,-n} \sim O(\varepsilon), \tag{27}$$

while the remaining eigenvalue pairs remain repeated. The plate response $w(\mathbf{r}, t)$ admits an eigenfunction representation

† ω_{mn} is normalized as $\omega_{mn} = \bar{\omega}_{mn}/\omega_{cr}$. $\bar{\omega}_{mn}$ is the eigenvalue corresponding to the eigenmode with m nodal circles and n nodal diameters of the axisymmetric plate, and ω_{cr} is the critical speed of the plate, i.e.

$$\omega_{cr} = \mathrm{Min}\left\{\frac{\bar{\omega}_{mn}}{n};\quad m = 0, 1, \ldots, n = 1, 2, \ldots\right\}.$$

The explicit expressions of $\phi_{mn}(\mathbf{r})$ are as follows. When $n = 0$, $\phi_{m0}(\mathbf{r}) = R_{m0}(r)$. When $n > 0$, $\phi_{mn}(\mathbf{r}) = R_{mn}(r)\cos(n\theta)$ and $\phi_{m,-n}(\mathbf{r}) = R_{mn}(r)\sin(n\theta)$. $R_{mn}(r)$ is a linear combination of Bessel's functions satisfying boundary conditions at both rims and the orthonormality condition

$$\frac{1}{b^2}\int_A \phi_{mn}(\mathbf{r})\phi_{pq}(\mathbf{r})\,\mathrm{d}A = \delta_{mp}\delta_{nq}$$

where A and b are the domain and outer radius of the plate, respectively. Similarly, β_{mn} is normalized as $\beta_{mn} = \bar{\beta}_{mn}/\omega_{cr}$, where $\bar{\beta}_{mn}$ is the asymmetric plate eigenvalue.

$$w(\mathbf{r},t)=\frac{1}{\sqrt{\rho_0 hb^2}}\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\psi_{mn}(\mathbf{r})q_{mn}(t) \tag{28}$$

ρ_0, h and b are density, thickness and rim radius of the plate, respectively. $t=\omega_{cr}\bar{t}$ is a normalized time, $\bar{t}$ is the physical time, and $q_{mn}(t)$ is a generalized coordinate.

When the asymmetric plate is excited by a spring rotating at normalized rotation speed $\Omega\equiv\bar{\Omega}/\omega_{cr}$ along a circle at $r=r_0$. Shen and Mote (1991b) showed that $\mathbf{q}$, the vector of $q_{mn}(t)$, satisfies (2) and (3a–c) in which $\mathbf{C}^{(0)}(\varepsilon)\sim O(\varepsilon)$ is positive definite, $\mathbf{B}(\varepsilon)$ is diagonal with elements β_{mn}^2, and

$$\mathbf{H}^{(s)}(\varepsilon)=\frac{1}{T}\int_0^T \mathrm{e}^{-is\Omega t}\boldsymbol{\psi}(\mathbf{r}_0,\Omega t)\boldsymbol{\psi}(\mathbf{r}_0,\Omega t)^{\mathrm{T}}\,\mathrm{d}t,\quad T=\frac{2\pi}{\Omega} \tag{29}$$

where $\boldsymbol{\psi}(\mathbf{r}_0,\Omega t)$ is the vector of $\psi_{mn}(\mathbf{r})$ evaluated at $r=r_0$ and $\theta=\Omega t$. Therefore, the perturbation analysis and the special cases in Sections 2 and 3 can be applied to predict the stability of the asymmetric plate/spring system.

4.2. *Primary instability*

Primary resonance of $O(\kappa)$, $(\kappa\ll\varepsilon)$. Let the inclusions be elastic and consider two vibration modes $\psi_{mn}(\mathbf{r})$ and $\psi_{ij}(\mathbf{r})$ with eigenvalues β_{mn} and β_{ij}. If $\psi_{mn}(\mathbf{r})$ and $\psi_{ij}(\mathbf{r})$ are split modes, then as specified in Section 2

$$\omega_i=\omega_{\mathbf{j}}=\beta_{mn},\quad \omega_k=\omega_{\mathbf{k}}=\beta_{ij},\quad \alpha=\beta=1,\quad \Delta_{\mathbf{j}}=\Delta_{\mathbf{k}}=0,\quad l=|n\pm j|\neq 0$$

because a pair of split modes $\psi_{mn}(\mathbf{r})$ and $\psi_{m,-n}(\mathbf{r})$ are not approximately repeated when $\kappa\ll\Delta\beta_{mn}$ [*cf.* (27)]. Substitute (26b) into (29), and recall the definition of $\mathbf{K}^{\langle 0\rangle}$ in (13f) and the explicit expressions of ϕ_{ii} to obtain

$$\mathbf{K}_{jj}^{\langle 0\rangle}=\frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}},\quad \mathbf{K}_{\mathbf{kk}}^{\langle 0\rangle}=\frac{|\gamma_{ij}^{\langle 0\rangle}|^2}{\beta_{ij}},\quad \mathbf{K}_{\mathbf{jk}(l)}^{\langle 0\rangle}=\frac{\gamma_{mn}^{(n)}\gamma_{ij}^{(\pm j)}}{2\sqrt{\beta_{mn}\beta_{ij}}},\quad \mathbf{K}^{\langle 0\rangle}\sim O(1) \tag{30}$$

where

$$\gamma_{mn}^{(n)}=\bar{\gamma}_{mn}^{(-n)}=\tfrac{1}{2}R_{mn}(r_0),\quad \gamma_{m,-n}^{(n)}=\bar{\gamma}_{m,-n}^{(-n)}=-\frac{i}{2}R_{mn}(r_0),\quad n>0.$$

According to Section 3.1, resonance occurs at

$$l\Omega=\beta_{mn}+\beta_{ij}+\sigma,\quad l=|n\pm j|\neq 0,\quad \sigma=\kappa\sigma^*+O(\kappa^2). \tag{31}$$

The width σ of the resonance is determined by the eigenvalue κz^* of $\mathbf{U}_1$ satisfying

$$\left(z^*-\frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}}\right)\left(z^*-\sigma^*+\frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\right)+\frac{|\gamma_{mn}^{(n)}\gamma_{ij}^{(\pm j)}|^2}{4\beta_{mn}\beta_{ij}}=0.$$

Therefore, the system is unstable when

$$\left|\sigma^*-\left(\frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}}+\frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\right)\right|\leqslant\frac{|\gamma_{mn}^{(n)}\gamma_{ij}^{(\pm j)}|}{\sqrt{\beta_{mn}\beta_{ij}}} \tag{32}$$

where the superscript $(\pm j)$ corresponds to instability associated with $l=|n\pm j|$.

If $\psi_{mn}(\mathbf{r})$ and $\psi_{m,-n}(\mathbf{r})$ are repeated modes and $\psi_{ij}(\mathbf{r})$ is a split mode, then a similar approach results in resonances (31) when

$$\left|\sigma^* - \left(\frac{|\gamma_{mn}^{(m)}|^2}{\beta_{mn}} + \frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\right)\right| \leq \frac{2|\gamma_{mn}^{(n)}\gamma_{ij}^{(\pm j)}|}{\sqrt{\beta_{mn}\beta_{ij}}}. \tag{33}$$

Similarly, if $\psi_{mn}(\mathbf{r})$, $\psi_{m,-n}(\mathbf{r})$ and $\psi_{ij}(\mathbf{r})$, $\psi_{i,-j}(\mathbf{r})$ are two pairs of distinct repeated modes, then the system is unstable when

$$\left|\sigma^* - \left(\frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}} + \frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\right)\right| \leq \frac{2|\gamma_{mn}^{(n)}\gamma_{ij}^{(\pm j)}|}{\sqrt{\beta_{mn}\beta_{ij}}}. \tag{34}$$

The threshold rotation speed of the primary resonances $\Omega_{th} = (\beta_{mn}+\beta_{ij})/|n\pm j|$ [*cf.* (31)] occurs at supercritical speed (i.e. $\Omega_{th} > 1$) and below the critical speed within $O(\varepsilon)$ [i.e. $\Omega_{th} - 1 \sim O(\varepsilon)$] as follows. Equation (26a) implies that

$$\Omega_{th} = \frac{\omega_{mn}+\omega_{ij}}{|n\pm j|} + \frac{\varepsilon}{|n\pm j|}\left[\frac{\mu_{mn}^{(1)}}{2\omega_{mn}} + \frac{\mu_{ij}^{(1)}}{2\omega_{ij}}\right] + O(\varepsilon^2)$$

in which

$$\frac{\omega_{mn}+\omega_{ij}}{|n\pm j|} \geq \mathrm{Min}\left(\frac{\omega_{mn}}{|n|}, \frac{\omega_{ij}}{|j|}\right) \geq 1.$$

Therefore, $\Omega_{th} > 1$ when the inequality holds, and $\Omega_{th} - 1 \sim O(\varepsilon)$ when the equality holds.

When the inclusions are viscoelastic, positive definite damping $\mathbf{C}^{(0)}(\varepsilon) \sim O(\varepsilon)$ suppresses all the resonances for $\kappa \ll \varepsilon$ according to Section 3.1. It can also be shown that (32) and (33) are valid when $\psi_{ij}(\mathbf{r})$ is an axisymmetric mode (i.e. $j = 0$) with $\gamma_{i0}^{(0)} = R_{i0}(r_0)$.

Primary resonance of $O(\varepsilon)$, $[\kappa \sim O(\varepsilon)]$. Let the inclusions be elastic, and consider the combination resonances caused by a pair of split modes $\psi_{mn}(\mathbf{r})$, $\psi_{m,-n}(\mathbf{r})$ and a pair of repeated modes $\psi_{ij}(\mathbf{r})$, $\psi_{i,-j}(\mathbf{r})$. Since the pair of split modes $\psi_{mn}(\mathbf{r})$ and $\psi_{m,-n}(\mathbf{r})$ are almost repeated because $\kappa \sim \Delta\beta_{mn}$, as specified in Section 2

$$\omega_j = \omega_{\mathrm{j}} = \beta_{mn}, \quad \omega_{j+1} = \beta_{m,-n}, \quad \omega_k = \omega_{k+1} = \omega_{\mathbf{k}} = \beta_{ij} \tag{35a}$$

$$\alpha = \beta = 2, \quad \Lambda_{\mathrm{j}} = \mathrm{diag}\,[0, \; -\Delta\beta_{mn}] \sim O(\varepsilon), \quad \mathbf{\Lambda}_{\mathbf{k}} = 0, \quad l = |n\pm j| \neq 0 \tag{35b}$$

with

$$\mathbf{K}_{\mathbf{jj}}^{(0)} = \frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}}\mathbf{I}_2, \quad \mathbf{K}_{\mathbf{kk}}^{(0)} = \frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\mathbf{I}_2, \quad \mathbf{K}_{\mathbf{jk}(l)}^{(0)} = \frac{\gamma_{mn}^{(n)}\gamma_{ij}^{(\pm j)}}{2\sqrt{\beta_{mn}\beta_{ij}}}\begin{bmatrix} 1 & \mp i \\ -i & \mp 1 \end{bmatrix}. \tag{36}$$

The instability zone is then determined in Section 3.2 with (35a,b) and (36). Unlike parametric resonances of $O(\kappa)$, a close form expression such as (33) cannot be obtained because $\mathbf{\Lambda}_{\mathrm{j}} \neq \mathbf{0}$.

When $\kappa \ll \varepsilon$, two separate primary resonances of $O(\kappa)$ occur as predicted by (33). When $\kappa \sim O(\varepsilon)$, only one primary resonance of $O(\varepsilon)$ occurs as predicted in Section 3.2. This implies a coalescence at $\kappa \sim O(\varepsilon)$ of two separate combination resonances emanating from

$$\Omega = \frac{\beta_{mn}+\beta_{ij}}{|n\pm j|} \quad \text{and} \quad \Omega = \frac{\beta_{m,-n}+\beta_{ij}}{|n\pm j|}.$$

This coalescence of instabilities occurs only when a pair(s) of split modes participate(s) in the instability. For instability not involving split modes, the results for $\kappa \ll \varepsilon$ in (31) to (34) apply for $\kappa \sim O(\varepsilon)$ because $\mathbf{U}_1 = \mathbf{U}_2$ in the undamped system. When damping is present, instability zones are determined by the eigenvalues κ_2^* of $\mathbf{U}_2$.

4.3. *Secondary instability*

Let the inclusions be elastic, and $\psi_{mn}(\mathbf{r})$ and $\psi_{ij}(\mathbf{r})$ be split modes of the plate. Consider

$$\omega_j = \omega_{\mathbf{j}} = \beta_{mn}, \quad \omega_k = \omega_{\mathbf{k}} = \beta_{ij}, \quad \alpha = \beta = 1, \quad \Delta_{\mathbf{j}} = \Delta_{\mathbf{k}} = 0, \quad l \neq |n \pm j|, \quad l \neq 0,$$

and note that

$$\mathbf{K}_{\mathbf{jj}}^{\langle 0 \rangle} = \frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}}, \quad \mathbf{K}_{\mathbf{kk}}^{\langle 0 \rangle} = \frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}, \quad \mathbf{K}_{\mathbf{jk}(l)}^{\langle 0 \rangle} \sim O(\varepsilon), \quad \mathbf{K}^{\langle 1 \rangle} \sim O(1) \tag{37}$$

for $l \neq |n \pm j|$ [*cf.* primary instability (30)]. According to Section 3.3, the instability is secondary and is centered with respect to a straight line

$$l\Omega = \kappa\left(\frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}} + \frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\right) + \beta_{mn} + \beta_{ij} \tag{38}$$

and the unstable zone width σ^{**} satisfies

$$\kappa^2\left|\sigma^{**} + \frac{1}{2\beta_{mn}}\left(\frac{|\gamma_{mn}^{(n)}|^2}{\beta_{mn}}\right)^2 + \frac{1}{2\beta_{ij}}\left(\frac{|\gamma_{ij}^{(j)}|^2}{\beta_{ij}}\right)^2 + tr\,\mathbf{K}^{\langle 1 \rangle}\right| \leqslant 2\sqrt{|\mathrm{Det}\,\mathbf{D}_4|} \tag{39}$$

where $tr\,\mathbf{K}^{\langle 1 \rangle}$ is the trace of $\mathbf{K}^{\langle 1 \rangle}$.

Secondary instabilities involving repeated modes or axisymmetric modes can be found in a similar manner. The resulting instability zones in the κ–Ω plane will also be centered with respect to (38). If damping $O(\varepsilon)$ is present, secondary instability will be suppressed for $\kappa \sim O(\varepsilon)$.

4.4. *Numerical examples*

Stability of the plate/spring system, studied numerically by Shen and Mote (1991b), is predicted by the perturbation method. The plate is a uniform Kirchhoff circular plate with three evenly spaced, radial inclusions. The plate is fixed at $r/b = 0.5$ and free at $r/b = 1$. The inclusions each span a small angle $\varepsilon = 0.035$ radian ($\approx 2°$), are located at $\theta = 0°$, $120°$ and $240°$, and extend from $r/b = 0.75$ to $r/b = 1$. The inclusions are elastic or viscoelastic. For elastic inclusions, the material properties satisfy

$$\frac{\rho_0'}{\rho_0} = \frac{E_0'}{E_0} = 0.5; \quad \sigma_0 = \sigma_0' = 0.3 \tag{40}$$

where ρ, E and σ are density, Young's modulus and Poisson's ratio. The primed quantities refer to the inclusions and the unprimed to the plate. For viscoelastic inclusions, the material damping satisfies

$$\xi = \frac{E_0^*}{E_0}\sqrt{\frac{E_0 h^2}{4\rho_0 b^4}} = 0.05; \quad \sigma_0^* = 0.3 \tag{41}$$

in addition to (40). Furthermore, the plate is subjected to a transverse spring rotating along the circle $r/b = 1$. A split mode pair $\psi_{0,-3}(\mathbf{r})$, $\psi_{03}(\mathbf{r})$ and an axisymmetric mode $\psi_{00}(\mathbf{r})$ are used to predict transition curves in the $\kappa-\Omega$ plane. The eigensolutions used are shown explicitly in Shen and Mote (1991b).

Figure 1 shows primary resonances when the inclusions are elastic. The solid lines are predictions from the perturbation analysis and the points are from numerical integration of (2). The perturbation solution results from (31) to (34) for $\kappa \ll \varepsilon$, and from (35a,b) to (36) for $\kappa \sim O(\varepsilon)$. These two parts are then matched at instability coalescence. In Fig. 1, three primary instability branches emanate from $\Omega = 0.998$, 1.002 and 1.007 representing the parametric resonance of $\psi_{0,-3}(\mathbf{r})$, combination resonance of the sum type for $\psi_{0,-3}(\mathbf{r})$ and $\psi_{03}(\mathbf{r})$ and the parametric resonance of $\psi_{03}(\mathbf{r})$ in that order. They coalesce when $\kappa > 0.08$. Similarly, two primary instabilities originating from $\Omega = 1.702$ and 1.711 are the combination resonances of $\psi_{00}(\mathbf{r})$, $\psi_{0,-3}(\mathbf{r})$ and $\psi_{00}(\mathbf{r})$, $\psi_{03}(\mathbf{r})$, respectively. They coalesce when $\kappa > 0.04$. The perturbation and the numerical integration predictions of the stability transition are in close agreement.

Figure 2 shows secondary resonances for elastic inclusions. The perturbation solution is predicted in Section 4.3. Both supercritical and subcritical unstable zones in the κ-Ω plane are found, and most of them are very narrow. Only the larger ones are shown in Fig. 2. Three secondary resonances from $\Omega \approx 2.0$ are from interaction between $\psi_{03}(\mathbf{r})$ and $\psi_{0,-3}(\mathbf{r})$. The one from $\Omega \approx 1.42$ is caused by $\psi_{00}(\mathbf{r})$. Two subcritical instabilities from $\Omega \approx 0.85$ are caused by $\psi_{00}(\mathbf{r})$, $\psi_{03}(\mathbf{r})$ and $\psi_{0,-3}(\mathbf{r})$. The perturbation solution deviates from the numerical one for $\kappa > 0.5$.

Figure 3 shows primary resonances for viscoelastic inclusions. Damping suppresses resonances for $\kappa \ll \varepsilon$ as predicted. Perturbation solution for $\kappa \sim O(\varepsilon)$ predicts the transition curves. The perturbation and the numerical solutions are in good agreement. Also notice that the unstable zone near $\Omega \approx 1.70$ is wider than it is in Fig. 1 because of disparate modal damping in $\psi_{03}(\mathbf{r})$, $\psi_{0,-3}(\mathbf{r})$ and $\psi_{00}(\mathbf{r})$.

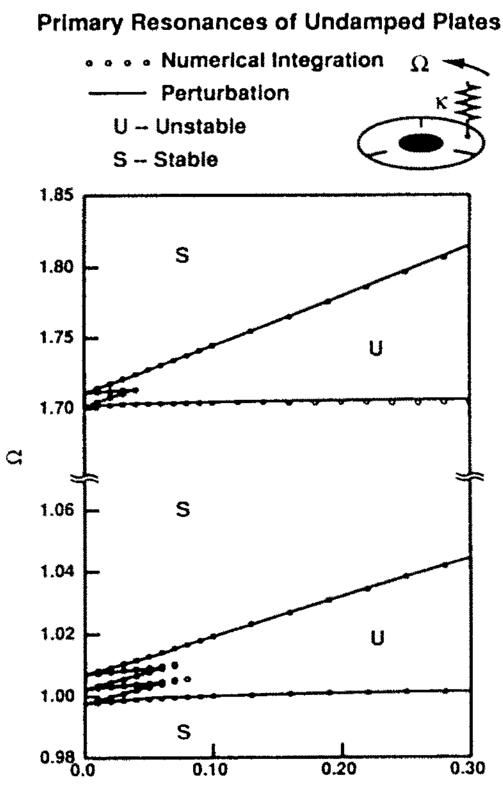

Fig. 1. Primary resonance stability boundary of a plate/spring system with three elastic inclusions predicted by three mode analysis $\psi_{0,-3}(\mathbf{r})$, $\psi_{03}(\mathbf{r})$ and $\psi_{00}(\mathbf{r})$.

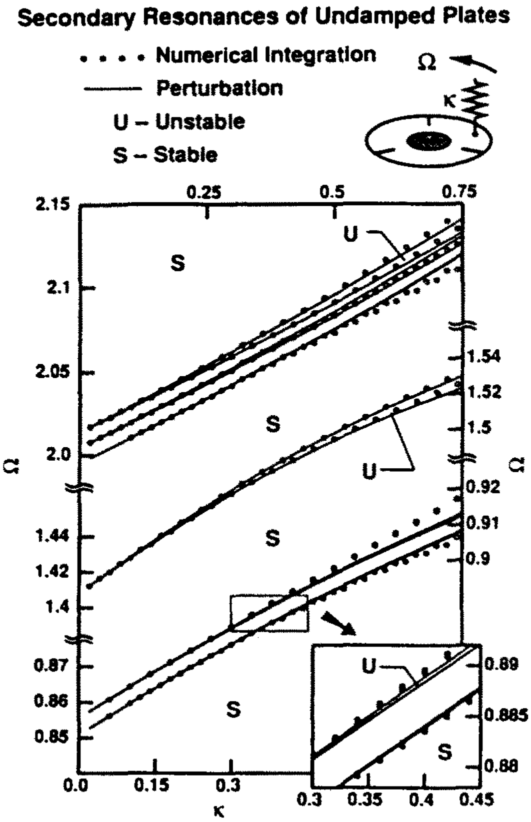

Fig. 2. Secondary resonance stability boundary of a plate/spring system with three elastic inclusions predicted by three mode analysis $\psi_{0,-3}(\mathbf{r})$, $\psi_{03}(\mathbf{r})$ and $\psi_{00}(\mathbf{r})$.

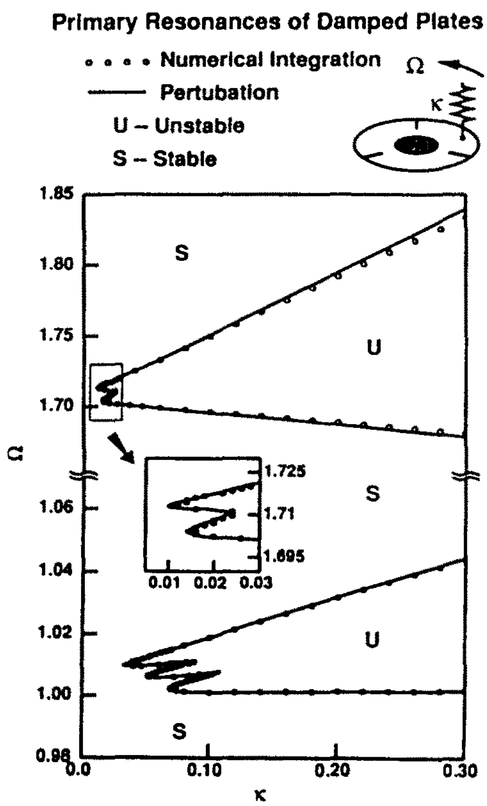

Fig. 3. Primary resonance stability boundary of a plate/spring system with three viscoelastic inclusions predicted by three mode analysis $\psi_{0,-3}(\mathbf{r})$, $\psi_{03}(\mathbf{r})$ and $\psi_{00}(\mathbf{r})$.

5. CONCLUSIONS

A perturbation method, based on the Floquet theorem and the method of successive approximations, is developed to determine the stability of parametrically excited systems containing multiple perturbation parameters. This method is illustrated by predicting the stability of a circular classical plate with elastic or viscoelastic inclusions subjected to a rotating transverse spring.

Primary resonances occur at supercritical speed or below critical speed within $O(\varepsilon)$. Their occurrence can be predicted analytically for $\kappa \ll \varepsilon$. Each split mode causes distinct instability zones for $\kappa \ll \varepsilon$. These zones coalesce when $\kappa \sim O(\varepsilon)$. Damping $O(\varepsilon)$ by the inclusions suppresses primary resonances for $\kappa \ll \varepsilon$.

Secondary resonance zones in the κ–Ω plalne are centered about straight lines predicted analytically by (38). Secondary resonances occur at subcritical speed for l sufficiently large though the unstable zones in the κ–Ω plane are often narrow. Damping $O(\varepsilon)$ in the inclusions suppresses secondary resonances for $\kappa \sim O(\varepsilon)$.

Acknowledgements—The authors would like to thank the National Science Foundation, the Computer Mechanics Laboratory and the authors' industrial sponsors for their support and interest in this work.

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